JOURNAL OF APPROXIMATION THEORY 91, 386–397 (1997)
ARTICLE NO. AT963093

Limit Theorems of Operators by Convex Combinations of Nonexpansive Retractions in Banach Spaces

Wataru Takahashi and Takayuki Tamura

Department of Information Science, Tokyo Institute of Technology, Tokyo 152, Japan

Communicated by Frank Deutsch

Received February 23, 1996; accepted in revised form October 15, 1996

1. INTRODUCTION

Let H be a Hilbert space, let $C_1, C_2, ..., C_r$ be nonempty closed convex subsets of H and let I be the identity operator on H. Then the convex feasibility problem in a Hilbert space setting may be stated as follows: The original (*unknown*) image z is known a priori to belong the intersection C_0 of r well-defined sets $C_1, C_2, ..., C_r$ in a Hilbert space H; given only the metric projections P_i of H onto C_i $(i=1, 2, ..., r)$, recover z by an iterative scheme.

In 1991, Crombez [4] proved the following: Let $T=\alpha_0 I+\sum_{i=1}^{r} \alpha_i T_i$ with $T_i = I+\lambda_i(P_i - I)$ for all i, $0<\lambda_i<2$, $\alpha_i>0$, $i=0, 1, 2, ..., r$, $\sum_{i=0}^{r} \alpha_i = 1$, where each P_i is the metric projection of H onto C_i and $C_0=\bigcap_{i=1}^{r} C_i$ is nonempty. Then starting from an arbitrary element x of H, the sequence $\{T^n x\}$ converges weakly to an element of C_0. But Crombez's result cannot be applied to this problem in L^p $(1<p<\infty, p\neq 2)$. Later, Kitahara and Takahashi [9] dealt with the convex feasibility problem by convex combinations of sunny nonexpansive retractions in uniformly convex Banach spaces. In [9], they proved that an operator given by a convex combination of sunny nonexpansive retractions in a uniformly convex Banach space is asymptotically regular and the set of fixed points of the operator is equal to the intersection of the ranges of sunny nonexpansive retractions. Further, using the results, they proved some weak convergence theorems for the operator which are connected with the convex feasibility problem. See also Reich [12].

In this paper, we also deal with the convex feasibility problem in Banach spaces setting and improve some results in [9]. We first prove two weak convergence theorems for an operator given by a convex combination of nonexpansive retractions in a strictly convex and reflexive Banach space. In the proofs of the theorems, it is crucial that the operator is asymptotically

0021-9045/97 $25.00

regular and the set of fixed points of the operator is equal to the intersection of ranges of nonexpansive retractions. One of the crucial results is proved using Edelstein and O'Brien [5] or Ishikawa [7] and the other is obtained using Bruck [1]. An important branch of the convex feasibility problem is the problem of image recovery. They often and seriously dealt with the problem of image recovery under the inconsistent constraints. So, we also pay attention to the situation where the constraints are inconsistent, i.e., when the intersection of the sets C_i $(i=1, 2, ..., r)$ is empty. Finally we consider the problem of finding a common fixed point for a finite commuting family of nonexpansive mappings in a strictly convex and reflexive Banach space.

2. PRELIMINARIES

Throughout this paper, we denote by **N** the set of positive integers and by **R** the set of real numbers. Let E be a Banach space and let I be the identity operator on E. Let C be a nonempty subset of E. Then, a mapping T of C into itself is said to be *nonexpansive* on C if $\|Tx - Ty\| \leqslant \|x - y\|$ for every $x, y \in C$. Let T be a mapping of C into itself. Then we denote by $F(T)$ the set of fixed points of T and by $R(T)$ the range of T. A mapping T of C into itself is said to be *asymptotically regular* if for every $x \in C$, $T^n x - T^{n+1} x$ converges to 0 as $n \to \infty$. Let D be a subset of C and let P be a mapping of C onto D. Then P is said to be *sunny* if

$$P(Px + t(x - Px)) = Px$$

whenever $Px + t(x - Px) \in C$ for $x \in C$ and $t \geqslant 0$. A mapping P of C into itself is said to be a *retraction* if $P = P^2$. If a mapping P of C into itself is a retraction, then $Pz = z$ for every $z \in R(P)$. A subset D of C is said to be a (sunny) nonexpansive retract if there exists a (sunny) nonexpansive retraction of C onto D. Let E be a Banach space and let $S_E = \{x \in E\colon \|x\| = 1\}$ be the unit sphere of E. Then, for every ε with $0 \leqslant \varepsilon \leqslant 2$, the modulus $\delta_E(\varepsilon)$ of convexity of a Banach space E is defined by

$$\delta_E(\varepsilon) = \inf \left\{ 1 - \frac{\|x + y\|}{2} \,\middle|\, \|x\| \leqslant 1,\ \|y\| \leqslant 1,\ \|x - y\| \geqslant \varepsilon \right\}.$$

A Banach space E is said to be uniformly convex if

$$\delta_E(\varepsilon) > 0$$

for every $\varepsilon > 0$. A Banach space E is said to be strictly convex if

$$\left\| \frac{x+y}{2} \right\| < 1$$

for $x, y \in S_E$ with $x \neq y$. A uniformly convex Banach space is strictly convex. In a strictly convex space, we also have that if

$$\|x\| = \|y\| = \|(1-\lambda)x + \lambda y\| \qquad \text{for} \quad x, y \in E;\ \lambda \in (0,1),$$

then $x = y$. A closed convex subset C of a Banach space E is said to have normal structure if for each bounded closed convex subset K of C which contains at least two points, there exists an element x_0 of K such that $\sup_{y \in K} \|x_0 - y\| < \delta(K)$, where $\delta(K)$ is the diameter of K. It is well known that a closed convex subset of a uniformly convex Banach space has normal structure and a compact convex subset of a Banach space has normal structure. The following result was proved by Kirk [8].

THEOREM 2.1 (Kirk [8]). *Let E be a reflexive Banach space and let C be a nonempty bounded closed convex subset of E which has normal structure. Let T be a nonexpansive mapping of C into itself. Then $F(T)$ is nonempty.*

Let E be a Banach space and let E^* be its dual, that is, the space of all continuous linear functionals f on E. Then the norm of E is said to be Gâteaux differentiable if

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t}$$

exists for each x and y in S_E. It is said to be Fréchet differentiable if for each x in S_E, this limit is attained uniformly for y in S_E. The following result is a direct consequence of Bruck [3]; see also [10, 15].

THEOREM 2.2 ([9]). *Let E be a uniformly convex Banach space with a Fréchet differentiable norm, and let C be a nonempty closed convex subset of E. Let T be an asymptotically regular nonexpansive mapping of C into itself with $F(T) \neq \emptyset$. Then, for each $x \in C$, $\{T^n x\}$ converges weakly to an element of $F(T)$.*

A Banach space E is said to satisfy Opial's condition [11] if $x_n \rightharpoonup x$ and $x \neq y$ imply

$$\liminf_{n \to \infty} \|x_n - x\| < \liminf_{n \to \infty} \|x_n - y\|,$$

where $\rightharpoonup$ denotes the weak convergence.

3. WEAK CONVERGENCE THEOREMS

In this section, we prove two weak convergence theorems which are connected with the convex feasibility problem in a Banach space setting. Using Edelstein and O'Brein [5] or Ishikawa [7], we first prove the following lemma.

LEMMA 3.1. *Let E be a Banach space and let C be a nonempty convex subset of E. Let S be a mapping on C given by $S=\beta_0 I+\sum_{i=1}^r \beta_i S_i$, $0<\beta_i<1$, $i=0, 1, ..., r$. $\sum_{i=0}^r \beta_i=1$, such that each S_i is nonexpansive on C and $\bigcap_{i=1}^r F(S_i)$ is nonempty. Then, S is asymptotically regular on C.*

Proof. Define a mapping T of C into itself by

$$Tx=\sum_{i=1}^r \frac{\beta_i}{1-\beta_0} S_i x \qquad \text{for every } x\in C.$$

Then T is nonexpansive. Further, since $\bigcap_{i=1}^r F(S_i)$ is nonempty, for any $x\in C$, $\{T^n x\}$ is bounded. So, from $S=\beta_0 I+(1-\beta_0)T$ and Theorem 1 in [5], we have that S is asymptotically regular on C. ∎

The following lemma proved by Bruck [1] is crucial in the proofs of Theorems 3.3 and 3.4. We give the proof for the sake of using it in the proof of Theorem 4.1.

LEMMA 3.2. *Let E be a strictly convex Banach space and let C be a nonempty closed convex subset of E. Let $C_1, C_2, ..., C_r$ be nonexpansive retracts of C such that $\bigcap_{i=1}^r C_i\neq\phi$. Let T be a mapping on C given by $T=\sum_{i=1}^r \alpha_i T_i$, $0<\alpha_i<1$, $i=1, 2, ..., r$, $\sum_{i=1}^r \alpha_i=1$, such that for each i, $T_i=(1-\lambda_i)I+\lambda_i P_i$, $0<\lambda_i<1$, where P_i is a nonexpansive retraction of C onto C_i. Then,*

$$F(T)=\bigcap_{i=1}^r C_i.$$

Proof. Let $x\in C_i$. Then, since P_i is a retraction of C onto C_i, there exists $y\in C$ with $P_i\, y=x$. So, we have $x=P_i\, y=P_i^2 y=P_i x$ and, hence, $T_i x=x$. Then $x\in F(T_i)$. It is obvious that $F(T_i)\subset C_i$. Therefore, $\bigcap_{i=1}^r C_i=\bigcap_{i=1}^r F(T_i)$. So, it is sufficient to show

$$F(T)=\bigcap_{i=1}^r C_i.$$

Let $x \in F(T)$. Then, for any $y \in \bigcap_{i=1}^{r} C_i$, we have

$$\begin{aligned}
\|x-y\| &= \|Tx-Ty\| \\
&= \left\| \sum_{i=1}^{r} \alpha_i T_i x - \sum_{i=1}^{r} \alpha_i T_i\, y \right\| \\
&= \left\| \sum_{i=1}^{r} \alpha_i (T_i x - T_i\, y) \right\| \\
&\leqslant \sum_{i=1}^{r} \alpha_i \, \|T_i x - T_i\, y\| \\
&= \sum_{i=1}^{r} \alpha_i \, \|(1-\lambda_i)\, x + \lambda_i P_i x - (1-\lambda_i)\, y - \lambda_i P_i\, y\| \\
&= \sum_{i=1}^{r} \alpha_i \, \|(1-\lambda_i)(x-y) + \lambda_i (P_i x - P_i\, y)\| \\
&= \sum_{i=1}^{r} \alpha_i \, \|(1-\lambda_i)(x-y) + \lambda_i (P_i x - y)\| \\
&\leqslant \sum_{i=1}^{r} \alpha_i ((1-\lambda_i)\, \|x-y\| + \lambda_i \, \|P_i x - y\|) \\
&\leqslant \sum_{i=1}^{r} \alpha_i ((1-\lambda_i)\, \|x-y\| + \lambda_i \, \|x - y\|) \\
&= \sum_{i=1}^{r} \alpha_i \, \|x-y\| \\
&= \|x-y\|.
\end{aligned}$$

So, we have, for each i,

$$\|x-y\| = \|P_i x - y\| = \|(1-\lambda_i)(x-y) + \lambda_i (P_i x - y)\|.$$

From strict convexity of E, we have $P_i x - y = x - y$ for each i. This implies $P_i x = x$ for each i. Therefore, $x \in \bigcap_{i=1}^{r} C_i$. ∎

Now we give the first weak convergence theorem for nonexpansive mappings given by convex combinations of retractions. This is a generalization of [9].

THEOREM 3.3. *Let E be a uniformly convex Banach space with a Fréchet differentiable norm and let C be a nonempty closed convex subset of E. Let $C_1, C_2, ..., C_r$ be nonexpansive retracts of C such that $\bigcap_{i=1}^{r} C_i \neq \phi$. Let T be a mapping on C given by $T = \sum_{i=1}^{r} \alpha_i T_i$, $0 < \alpha_i < 1$, $i = 1, ..., r$, $\sum_{i=1}^{r} \alpha_i = 1$,* *such that for each i,* $T_i = (1-\lambda_i) I + \lambda_i P_i$, $0 < \lambda_i < 1$, *where* P_i *is a nonexpansive retraction of* C *onto* C_i. *Then,* $F(T) = \bigcap_{i=1}^r C_i$ *and, further, for each* $x \in C$, $\{T^n x\}$ *converges weakly to an element of* $\bigcap_{i=1}^r C_i$.

Proof. Since E is uniformly convex, E is strictly convex. So, we have $F(T) = \bigcap_{i=1}^r F(T_i) = \bigcap_{i=1}^r C_i$ by Lemma 3.2. As in the proof of Theorem 6 in [9], T is asymptotically regular on C. So, it follows from Theorem 2.2 that for each $x \in C$, $\{T^n x\}$ converges weakly to an element of $F(T) = \bigcap_{i=1}^r C_i$. ▮

Further we have following.

THEOREM 3.4. *Let* E *be a reflexive and strictly convex Banach space satisfying Opial's condition and let* C *be a nonempty closed convex subset of* E. *Let* $C_1, C_2, ..., C_r$ *be nonexpansive retracts of* C *such that* $\bigcap_{i=1}^r C_i \neq \phi$. *Let* T *be a mapping on* C *given by* $T = \sum_{i=1}^r \alpha_i T_i$, $0 < \alpha_i < 1$, $i = 1, ..., r$, $\sum_{i=1}^r \alpha_i = 1$, *such that for each i,* $T_i = (1-\lambda_i) I + \lambda_i P_i$, $0 < \lambda_i < 1$, *where* P_i *is a nonexpansive retraction of* C *onto* C_i. *Then,* $F(T) = \bigcap_{i=1}^r C_i$ *and, further, for each* $x \in C$, $\{T^n x\}$ *converges weakly to an element of* $\bigcap_{i=1}^r C_i$.

Proof. As in the proof of Theorem 3.3, it follows that $F(T) = \bigcap_{i=1}^r C_i$ and T is asymptotically regular on C. So, we show that for any $x \in C$, $\{T^n x\}$ converges weakly to an element of $\bigcap_{i=1}^r C_i$. Let $x \in C$. Since $F(T)$ is nonempty, $\{T^n x\}$ is bounded. Then, since E is reflexive, there exists a subsequence $\{T^{n_i} x\}$ of $\{T^n x\}$ converging weakly to an element z of C. To complete the proof of Theorem 3.4, it is sufficient to prove that $z \in \bigcap_{i=1}^r C_i$ and if another subsequence $\{T^{n_j} x\}$ of $\{T^n x\}$ converging weakly to an element z', then $z = z'$. First, we prove $z \in F(T) = \bigcap_{i=1}^r C_i$. We assume $z \neq Tz$. Since T is asymptotically regular on C, we also have that $\{T^{n_i+1} x\}$ converges weakly to z. Further, since E satisfies Opial's condition, then we have

$$\begin{aligned}
\liminf_i \|T^{n_i} x - z\| &\leqslant \liminf_i (\|T^{n_i} x - T^{n_i+1} x\| + \|T^{n_i+1} x - z\|) \\
&= \liminf_i \|T^{n_i+1} x - z\| \\
&< \liminf_i \|T^{n_i+1} x - Tz\| \\
&\leqslant \liminf_i \|T^{n_i} x - z\|.
\end{aligned}$$

It is a contradiction. So, we have $z \in F(T)$. Similarly, we have $z' \in F(T)$. Since T is nonexpansive, limits of $\|T^n x - z\|$ and $\|T^n x - z'\|$ exist. Now we show $z = z'$. We assume $z \neq z'$. Then we have

$$\begin{aligned}\liminf_i \|T^{n_i}x - z\| &< \liminf_i \|T^{n_i}x - z'\| \\ &= \lim_n \|T^n x - z'\| \\ &= \liminf_j \|T^{n_j}x - z'\| \\ &< \liminf_j \|T^{n_j}x - z\| \\ &= \lim_n \|T^n x - z\| \\ &= \liminf_i \|T^{n_i}x - z\|.\end{aligned}$$

This is a contradiction. So, we have $z = z'$. This completes the proof. ▮

4. ADDITIONAL RESULTS

In this section, we first consider the convex feasibility problem under the situation where the constraints are inconsistent. Then, we consider the problem of finding a common fixed point for a finite commuting family of nonexpansive mappings. Let μ be a mean on $\mathbf{N}$, i.e., a continuous linear functional on l_∞ satisfying $\|\mu\| = 1 = \mu(1)$. We know that μ is a mean on $\mathbf{N}$ if and only if

$$\inf\{a_n : n \in \mathbf{N}\} \leqslant \mu(a) \leqslant \sup\{a_n : n \in \mathbf{N}\}$$

for every $a = (a_1, a_2, ...) \in l_\infty$. Occasionally, we use $\mu_n(a_n)$ instead of $\mu(a)$. So, a Banach limit μ is a mean μ on $\mathbf{N}$ satisfying $\mu_n(a_n) = \mu_n(a_{n+1})$.

LEMMA 4.1. *Let E be a reflexive Banach space and let C be a nonempty closed convex subset of E which has the normal structure. Let $C_1, C_2, ..., C_r$ be nonempty bounded nonexpansive retracts of C. Let T be a mapping on C given by $T = \sum_{i=1}^r \alpha_i T_i$, $0 < \alpha_i < 1$, $i = 1, ..., r$, $\sum_{i=1}^r \alpha_i = 1$, such that for each i, $T_i = (1 - \lambda_i) I + \lambda_i P_i$, $0 < \lambda_i < 1$, where P_i is a nonexpansive retraction of C onto C_i. Then $F(T)$ is nonempty. Further, assume that E is strictly convex and $\bigcap_{i=1}^r C_i = \phi$. Then $F(T) \cap C_i = \phi$ for some i.*

Proof. Let $x \in C$ and consider a closed ball $B_R[x]$ of center x and radius R containing all the sets $C_1, C_2, ..., C_r$. Then we have $\{T^n x\} \subset B_R[x] \cap C$. This implies that $\{T^n x\}$ is bounded. So, we define a real valued function g on C by

$$g(y) = \mu_n \|T^n x - y\| \qquad \text{for every} \quad y \in C,$$

where μ is a Banach limit on l_∞ and set

$$M = \{z \in C : \mu_n \|T^n x - z\| = \inf_{y \in C} \mu_n \|T^n x - y\|\}.$$

Then M is nonempty, bounded, closed, and convex. Further, M is invariant under T; for more details see [9, 12]. So, since T is nonexpansive, by Theorem 1, we have a fixed point of T in M. Assume $\bigcap_{i=1}^{r} C_i = \phi$ and let $x, y \in F(T)$. Then we have

$$x = \sum_{i=1}^{r} \alpha_i \{(1 - \lambda_i) x + \lambda_i P_i x\}$$

and

$$y = \sum_{i=1}^{r} \alpha_i \{(1 - \lambda_i) y + \lambda_i P_i y\}.$$

So, we obtain, as in the proof of Lemma 3.2,

$$\begin{aligned} \|x - y\| &\leqslant \sum_{i=1}^{r} \alpha_i \|(1 - \lambda_i)(x - y) + \lambda_i (P_i x - P_i y)\| \\ &\leqslant \sum_{i=1}^{r} \alpha_i \{(1 - \lambda_i) \|x - y\| + \lambda_i \|P_i x - P_i y\|\} \\ &\leqslant \|x - y\| \end{aligned} \tag{1}$$

and, hence,

$$\|x - y\| = \|P_i x - P_i y\| = \|(1 - \lambda_i)(x - y) + \lambda_i (P_i x - P_i y)\|$$

for each i. Since E is strictly convex, we have

$$x - y = P_i x - P_i y \tag{$*$}$$

for each i. Assume $F(T) \cap C_i \neq \phi$. Then we have $F(T) \subset C_i$. In fact, if $x \in F(T)$ and $y \in F(T) \cap C_i$, by $(*)$ we have

$$x - P_i x = y - P_i y = y - y = 0$$

and, hence, $x \in C_i$. Therefore $F(T) \subset C_i$. If $F(T) \cap C_i \neq \phi$ for every i, we have $F(T) \subset \bigcap_{i=1}^{r} C_i$. This contradicts $\bigcap_{i=1}^{r} C_i = \phi$. Therefore $F(T) \cap C_i = \phi$ for some i. ∎

Let C and D be nonempty convex subsets of a Banach space E. Then we denote by $i_C D$ the set of $z \in D$ such that for any $x \in C$, there exists $\lambda \in (0, 1)$

with $\lambda x+(1-\lambda) z \in D$ and by $\partial_C D$ the set of $z \in D$ such that there exists $x \in C$ with $\lambda x+(1-\lambda) z \notin D$ for all $\lambda \in (0, 1)$.

THEOREM 4.2. *Let E be a strictly convex and reflexive Banach space and let C be a nonempty closed convex subset of E which has normal structure. Let $C_1, C_2, ..., C_r$ be nonempty bounded sunny nonexpansive retracts of C such that for each i, an element of $\partial_C C_i$ is an extreme point of C_i. Let T be a mapping on C given by $T=\sum_{i=1}^{r} \alpha_i T_i$, $0<\alpha_i<1$, $i=1, ..., r$, $\sum_{i=1}^{r} \alpha_i=1$, such that for each i, $T_i=(1-\lambda_i) I+\lambda_i P_i$, $0<\lambda_i<1$, where P_i is a sunny nonexpansive retraction of C onto C_i.* If $\bigcap_{i=1}^{r} C_i$ *is empty, then $F(T)$ consists of one point. In addition, if E is uniformly convex or satisfies Opial's condition, for each $x \in C$, $\{T^n x\}$ converges weakly to an element of $F(T)$.*

Proof. By strict convexity of E and Lemma 4.1, $F(T)$ is a nonempty closed convex subset of C and $F(T) \cap C_j=\phi$ for some j. Let $u, v \in F(T)$. Then as in the proof of Lemma 4.1, we have $u-P_j u=v-P_j v$. So, for any $x, y \in F(T)$ and $\lambda \in (0, 1)$, we have $\lambda x+(1-\lambda) y \in F(T)$ and

$$\begin{aligned} &\|P_j(\lambda x+(1-\lambda) y)-(\lambda P_j x+(1-\lambda) P_j y)\| \\ &\quad =\|P_j(\lambda x+(1-\lambda) y)-\{\lambda x+(1-\lambda) y\}+\lambda x+(1-\lambda) y \\ &\qquad -(\lambda P_j x+(1-\lambda) P_j y)\| \\ &\quad =\|P_j x-x+\lambda(x-P_j x)+(1-\lambda)(y-P_j y)\| \\ &\quad =0. \end{aligned}$$

This implies that P_j is an one-to-one affine mapping of $F(T)$ onto C_j. Further, for any $x \in F(T)$, $P_j x \in \partial_C C_j$. In fact, if $P_j x \in i_C C_j$, there exists $\lambda \in (0, 1)$ with $\lambda x+(1-\lambda) P_j x \in C_j$. Since P_j is sunny, we have

$$\lambda x+(1-\lambda) P_j x=P_j(\lambda x+(1-\lambda) P_j x)=P_j x$$

and, hence, $x=P_j x$. This is a contradiction. Let $x, y \in F(T)$ with $x \neq y$. Then $P_j x \neq P_j y$ and for any $\lambda \in (0, 1)$,

$$P_j(\lambda x+(1-\lambda) y)=\lambda P_j x+(1-\lambda) P_j y.$$

This contradicts that $P_j(\lambda x+(1-\lambda) y)$ is an extreme point of C_j. Therefore $F(T)$ consists of one point. We assume that E is uniformly convex or Opial's condition. Let $x \in C$. Then since $I-T$ is demiclosed and T is asymptotically regular. $\{T^{n_i} x\} \rightharpoonup z$ implies $z \in F(T)$. By the uniqueness of $F(T)$, any weakly convergent subsequence $\{T^{n_i} x\}$ of $\{T^n x\}$ has the same weakly limit point. Then we have that, for each $x \in C$, $\{T^n x\}$ converges weakly to an element of $F(T)$. ∎

The following theorem related to the existence of a nonexpansive retract is proved in Bruck [1, 2]. See [9] for the existence of a sunny nonexpansive retract.

THEOREM 4.3. *Let E be a reflexive Banach space. Let C be a nonempty closed convex subset of E and let T be a nonexpansive mapping of C into itself with $F(T) \neq \phi$. If T has a fixed point in every nonempty bounded closed convex set that T leaves invariant, then $F(T)$ is a nonexpansive retract of C.*

Using Theorem 4.3, we prove the following.

THEOREM 4.4. *Let E be a uniformly convex Banach space with a Fréchet differentiable norm and let C be a nonempty closed convex subset of E. Let $\{S_1, S_2, ..., S_r\}$ be a commuting family of nonexpansive mappings on C with $F(S_i) \neq \phi$, $i = 1, 2, ..., r$. Let T be a mapping on C given by $T = \sum_{i=1}^{r} \alpha_i T_i$, $0 < \alpha_i < 1$, $i = 1, ..., r$, $\sum_{i=1}^{r} \alpha_i = 1$, such that for each i, $T_i = (1 - \lambda_i) I + \lambda_i P_i$, $0 < \lambda_i < 1$, where P_i is a nonexpansive retraction of C onto $F(S_i)$. Then, $F(T) = \bigcap_{i=1}^{r} F(S_i)$. Further, for each $x \in C$, $\{T^n x\}$ converges weakly to an element of $\bigcap_{i=1}^{r} F(S_i)$.*

Proof. Since E is uniformly convex, it follows from Theorem 2.1 that for each i, S_i has a fixed point in every nonempty bounded closed convex set that S_i leaves invariant. So, by Theorem 4.3, $F(S_i)$ is a nonexpansive retract of C for each i. However, as in the proof of Theorem 2 in [6], we show the existence of a nonexpansive retraction of C onto $F(S_i)$ without using Theorem 4.3. Let $x \in C$ and let μ be a Banach limit on l_∞. Then, for each S_i, define a function g of E^* into $\mathbf{R}$ by

$$g(x^*) = \mu_n \langle S_i^n x, x^* \rangle \qquad \text{for every} \quad x^* \in E^*.$$

Then g is linear and continuous. So, we have a unique element $x_0 \in E$ such that

$$\mu_n \langle S_i^n x, x^* \rangle = \langle x_0, x^* \rangle \qquad \text{for every} \quad x^* \in E^*.$$

Thus, putting $x_0 = P_i x$ for every $x \in C$, by [6] P_i is a nonexpansive retraction of C onto $F(S_i)$. Since E is strictly convex, $F(S_i)$ is nonempty, closed, and convex. By mathematical induction, we show that $\bigcap_{i=1}^{r} F(S_i)$ is nonempty. Let $z \in F(S_i)$. Then, from $S_i S_j z = S_j S_i z = S_j z$, we have $S_j z \in F(S_i)$. So, $F(S_i)$ is invariant under S_j. Let $u \in F(S_j)$. Then since E is reflexive and strictly convex, there exists a unique element $z \in F(S_i)$ such that $\|u - z\| = \min\{\|u - v\| : v \in F(S_i)\}$. For such a $z \in F(S_i)$, we have $\|S_j z - u\| \leqslant \|z - u\|$ and $S_j z \in F(S_i)$. So, we have $z = S_j z$. This implies that $F(S_i) \cap F(S_j) \neq \phi$. See, for more details, [9]. Therefore, by Lemma 3.2 and Theorem 3.3, we have that $F(T)=\bigcap_{i=1}^{r} F(S_i)$ and for each $x \in C$, $\{T^n x\}$ converges weakly to an element of $\bigcap_{i=1}^{r} F(S_i)$. ∎

THEOREM 4.5. *Let E be a reflexive and strictly convex Banach space which satisfies Opial's condition and let C be a nonempty closed convex subset of E. Let $\{S_1, S_2, ..., S_r\}$ be a commuting family of nonexpansive mappings on C such that $F(S_i) \neq \phi$ for $i=1, 2, ..., r$. Let T be a mapping on C given by $T=\sum_{i=1}^{r} \alpha_i T_i$, $0<\alpha_i<1$, $i=1, ..., r$, $\sum_{i=1}^{r} \alpha_i=1$, such that for each i, $T_i=(1-\lambda_i) I+\lambda_i P_i$, $0<\lambda_i<1$, where P_i is a nonexpansive retraction of C onto $F(S_i)$. Then, $F(T)=\bigcap_{i=1}^{r} F(S_i)$ and, further, for each $x \in C$, $\{T^n x\}$ converges weakly to an element of $\bigcap_{i=1}^{r} F(S_i)$.*

Proof. Fix i with $1 \leqslant i \leqslant r$ and let D be a nonempty bounded closed convex subset of C with $S_i D \subset D$. Then for any $u \in F(S_i)$, there exists a unique element $z \in D$ such that $\|u-z\| = \min\{\|u-v\| : v \in D\}$. Since D is invariant under S_i, we have $S_i z \in D$. Further, since $\|S_i z-u\| \leqslant \|z-u\|$, we have $S_i z=z$. This implies S_i has a fixed point in D. Then, by Theorem 4.3, $F(S_i)$ is a nonexpansive retract of C for each i. So, there exists a nonexpansive retraction P_i of C onto $F(S_i)$. As in the proof of Theorem 4.4, we have that $\bigcap_{i=1}^{r} F(S_i)$ is nonempty. By Lemma 3.2, we also have $F(T)=\bigcap_{i=1}^{r} F(S_i)$. Further, by Theorem 3.4, for each $x \in C$, $\{T^n x\}$ converges weakly to an element of $\bigcap_{i=1}^{r} F(S_i)$. ∎

ACKNOWLEDGMENTS

The authors express hearty thanks to the refereee for his careful reading of the manuscript and his valuable comments.

REFERENCES

1. R. E. Bruck, Properties of fixed-point sets of nonexpansive mappings in Banach spaces, *Trans. Amer. Math. Soc.* **179** (1973), 251–262.
2. R. E. Bruck, A common fixed point theorem for a commuting family of nonexpansive mappings, *Pacific J. Math.* **53** (1974), 59–71.
3. R. E. Bruck, A simple proof of the mean ergodic theorem for nonlinear contractions in Banach spaces, *Israel J. Math.* **32** (1979), 107–116.
4. G. Crombez, Image recovery by convex combinations of projections, *J. Math. Anal. Appl.* **155** (1991), 413–419.
5. M. Edelstein and R. C. O'Brien, Nonexpansive mappings, asymptotic regularity and successive approximations, *J. London Math. Soc.* **17** (1978), 547–554.
6. N. Hirano, K. Kido, and W. Takahashi, The existence of nonexpansive retractions in Banach spaces, *J. Math. Soc. Japan* **38** (1986), 1–7.
7. S. Ishikawa, Fixed points and iteration of a nonexpansive mapping in a Banach space, *Proc. Amer. Math. Soc.* **59** (1976), 65–71.

8. W. A. Kirk, A fixed point theorem for mappings which do not increase distances, *Amer. Math. Monthly* **72** (1965), 1004–1006.
9. S. Kitahara and W. Takahashi, Image recovery by convex combinations of sunny nonexpansive retractions, *Topol. Methods Nonlinear Anal.* **2** (1993), 333–342.
10. A. T. Lau and W. Takahashi, Weak convergence and non-linear ergodic theorems for reversible semigroups of nonexpansive mappings, *Pacific J. Math.* **126** (1987), 277–294.
11. Z. Opial, Weak convergence of the sequence of successive approximations for nonexpansive mappings, *Bull. Amer. Math. Soc.* **73** (1967), 591–597.
12. S. Reich, A limit theorem of projections, *Linear Multilinear Algebra* **13** (1983), 281–290.
13. W. Takahashi, Fixed point theorems for families of nonexpansive mappings on unbounded sets, *J. Math. Soc. Japan* **36** (1984), 543–553.
14. W. Takahashi, "Nonlinear Functional Analysis," Kindai–kagakusha, Tokyo, 1988. [Japanese]
15. W. Takahashi and J. Y. Park, On the asymptotic behavior of almost orbits of commutative semigroups in Banach spaces, *in* "Nonlinear and Convex Analysis" (B. L. Lin and S. Simons, Eds.), pp. 271–293, Lecture Notes in Pure and Appl. Math., Dekker, New York, 1987.